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A CONSTRUCTION FOR PARTIALLY ORDERED SETS

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A construction $I(S)$ is defined which corresponds to the intuitive notion of the set of places in which new elements can be inserted into a given poset S. It is given the minimal possible ordering. It turns out that if the base sets are chains the construction produces the corresponding interval orders, for whose dimensions there exist good estimates. In this paper we make the dual restriction that the height of the underlying set is $\leqslant 1$. Under this assumption we find a bound for the dimension of $I(S)$ in general and a precise value if the set consists of two antichains all the elements of one lying above all those of the other.

The construction that is the subject of this paper was inspired by Rabinovitch's definition of an interval order [5]. The idea is to extend a given poset S by adjoining one element in each "location" relative to the elements of S. The ordering between these elements is not completely determined by their location and in a previous paper [4] I investigated the maximal possible ordering, calling the poset thus defined the universal extension $U(S)$. In this paper we examine the minimal possible ordering, which we call $I(S)$ the interval extension of S. The set $U(S)$ is universal in a category theoretical sense and is "smoother" than S, for instance, its dimension is exactly the maximum cardinality of certain subsets of S (see [4]). The subsets conserned are described in this paper after Theorem 7. In contrast $I(S)$ is more complicated than S. However, it seems to contain as subsets many constructions investigated in the literature. In Section 1 we shall show this in detail for the generic example of an interval order $I(0, n)$ investigated in [1], we only mention here that it also contains Trotter's constructions in [7] and [8] and seems to be closely connected with such questions as the interval dimension of a set discussed in [2].

One of the difficulties in discussing the dimension of a poset is describing the constructions used to determine it. Section 2 introduces some terminology which allows us to clarify the ideas behind the most common constructions. This terminology is then applied to $I(S)$ to produce a necessary condition for a construction to be successful. An example is given to show that the condition is not sufficient.

In section three we consider sets of height 1. For these the condition is sufficient and this allows us to calculate the precise dimension of $I(S)$ if S is an antichain, of two antichains, one above the other. In a sense this calculation is dual to that performed by Bogart, Rabinovitch and Trotter in [1].

1. The construction

Definition. Let S be a poset. A *location* in S is a pair (D, U), where $D \subseteq S$ is a down set and $U \subseteq S$ is an up set, and every element of D lies below every element of U. (A down set D satisfies the condition that $s \in D$ and $t < s \Rightarrow t \in D$, an up set is defined dually). The set D is to be considered as the elements of S below the location and U as the set above the location. Thus any element of an extension of S either lies in S itself or defines a location in S namely (D, U) with $D = \{s \in S \mid s < x\}$, $U = \{s \in S \mid s > x\}$. We denote the set of locations in S by $L(S)$.

Let $X = S \cup L(S)$; from the above it is clear that a reasonable ordering of X must restrict to the given ordering on S and place $x = (D, U)$ above the elements of D, below those of U and leave it incomparable to any other elements of S. This leaves some leeway in choosing the ordering but only in its restriction to $L(S)$. We demarcate this leeway in the following proposition.

Proposition 1. *Let X be as above and "<" a reasonable ordering of X in the sense just discussed. For elements $x = (D, U)$, $y = (D', U') \in L(S)$ we have*

$$U \cap D' \neq \emptyset \Rightarrow x < y \Rightarrow D \subseteq D' \quad \text{and} \quad U \supseteq U'.$$

Definition. The interval extension $I(S)$ of S has underlying set $S \cup L(S)$ and is given the following ordering. Let $s, t \in S$, $x = (D, U)$, $y = (D', U') \in L(S)$, then

(a) $s < t$ in $I(S) \Leftrightarrow s < t$ in S,

(b) $s < x \Leftrightarrow s \in D$, $\quad s > x \Leftrightarrow s \in U$,

(c) $x < y \Leftrightarrow U \cap D' \neq \emptyset$.

The generic example of an interval order in the sense of Rabinovitch is the set $I[0, n]$ defined below. This set originally arose in the discussion of the ranking of elements whose position on some scale is only known approximately and is the subject of an important paper by Bogart, Rabinovitch and Trotter [1]. The set $I[0, n]$ consists of all closed intervals $[i, j]$ where i and j are integers with $0 \leqslant i \leqslant j \leqslant n$. The ordering of these intervals is $[i, j] < [k, l] \Leftrightarrow j < k$. We can identify this set with $L(C_n)$ where C_n is the chain $(1, \ldots, n)$ by equating $[i, j]$ with (D, U): $D = \{c \in C \mid c \leqslant i\}$, $U = \{c \in C \mid c > j\}$. It is clear that the ordering is the same as that defined in $I(C_n)$. The fact that C_n itself is omitted makes no difference, at least as far as the dimension is concerned, as the next proposition shows.

Before we state and prove the proposition we introduce some terminology from [4].

Definition. Let S be a poset, an *order R on S* is a partial ordering of the set S such that $a \leqslant b$ in S implies aRb. We shall write $a \leqslant b$ (R) for aRb, and denote the original ordering of S by the letter S itself. Note that we exclude relations that identify elements of S in order to avoid having to state this condition repeatedly.

In [4] such orders were not excluded. If $\{R_i\}$ is a set of orders on S the order $\bigcap R_i$ on S is given by

$$s<t(\bigcap R_i) \Leftrightarrow s<t(R_i) \quad \text{for all } i.$$

A set of linear orders $\{R_i\}$ on S is called a *base* if $S=\bigcap R_i$, that is

$$s<t(S) \Leftrightarrow s<t(R_i) \quad \text{for all } i.$$

The *dimension* of S is the minimal cardinality of a base on S.

The following statements are trivial. Any order on S can be extended to a linear order on S. Any order on a subset $T\subseteq S$ can be extended to an order on S that restricts to the given order on T. Finally if $\{R_i\}$ is a set of orders on S such that $s\nleq t(S) \Rightarrow s>t(R_i)$ for some R_i, then any set $\{R_i'\}$ where R_i' is a linear extension of R_i forms a base on S.

Proposition 2. *Let $L(S)$ be ordered as a subset of $I(S)$. Then* $\dim L(S)=\dim I(S)$.

Proof. Let $\{R_i\}$ be a base on $L(S)$. Extend each R_i to a linear order R_i' on $I(S)$. We claim $\{R_i'\}$ is a base on $I(S)$. For let $x\nleq y$ in $I(S)$. If both x and y are in $L(S)$, then there exists R_i with $x>y(R_i)$ and hence $x>y(R_i')$. Assume $x\in S$, $y\in L(S)$. Put $z=(D,\emptyset)$, $D=\{t\in S \mid t\leq x\}$. Then $x<z$ $(I(S))$ and $z\nless y$ in $L(S)$. However $x<z$, $x\nleq y$ implies also $z\neq y$. So there exists R_i such that $y>z(R_i)$. But then $y>x(R_i')$. The dual argument works if $x\in L(S)$, $y\in S$. So we are left with the case $x, y\in S$. Define z as before. Now $z\nleq y(I(S))$ so by the dual argument there exists R_i' with $y>z$ (R_i'). But again $x<z(I(S))$ so $y>x(R_i')$. Thus $\dim I(S)\leq \dim L(S)$. But $L(S)\subseteq I(S)$, so they must have the same dimension.

2. The construction of bases

The natural way of constructing a base of a poset is to start with a base of some subset Y of X and extend this to X by first placing the elements of $X\setminus Y$ with respect to those of Y and then ordering the elements that have been placed in the same slot. Of course it may be necessary to use a given order on Y several times in this process. We begin by introducing some terminology to formalize this definition, and then consider the case where $Y=S$, $X=I(S)$. We prove a generalization of the crucial property of $I[0,n]$ used by Rabinovitch [5] and derive a necessary condition for a set of orders on S to have extensions which form a base on $I(S)$. An example is given to show that this property is not sufficient.

Definition. Let $Y\subseteq X$ be posets, and R a linear order on Y. An R-*slot* in Y is a location (D,U) of R such that $D\cup U=Y$. If $x\in X\setminus Y$ the slot (D,U) is *admissible for* x if for $y \in Y$, $y < x(X) \Rightarrow y \in D$, $y > x(X) \Rightarrow y \in U$. We extend this definition to orderings R' of X that restrict to linear orders on Y and say that R' *places* x in the slot (D, U) if $D = \{y \in Y \mid y < x(R')\}$ and $U = \{y \in Y \mid x < y(R')\}$. We note that the R-slots in Y are linearly ordered and that if R' places x in the slot (D, U), then (D, U) is necessarily admissible for x.

We now restrict our attention to the case where $Y = S$, $X = I(S)$. It is then obvious that for any linear order R on S, $x \lessdot y$, x, $y \in L(S)$ implies that the admissible slots of x lie below the admissible slots of y. Thus if R' is any extension of R to $I(S)$ the set of elements $x \in L(S)$ placed in a given slot $a = (D, U)$ from an antichain in $I(S)$. We can now derive the property of $I(S)$ that generalizes Rabinovitch's crucial property of $I[0, n]$ see [1, p. 326]. The property states that in extending a linear order R on S to $I(S)$ the choice of admissible slot is free and so is the ordering of the elements of $L(S)$ placed in a given slot.

Lemma 3. *Let R be a linear order on S. To each $x \in L(S)$ choose an admissible slot $a(x)$ in R. To each slot a of R choose an arbitrary ordering Q_a of the elements $x \in L(S)$ with $a(x) = a$. Then there exists an extension R' of R to $I(S)$ placing each element x of $L(x)$ in the chosen admissible slot $a(x)$, and ordering the elements placed in a by Q_a.*

We omit the straightforward proof.

The next proposition shows that in constructing a base on $I(S)$ the difficult part is placing locations in slots. Once that has been done, suitable orderings for the elements in each slot can always be found.

Proposition 4. *Let $\{R_i\}$, $i = 1, \ldots, t$ be a set of at least two linear orders on S (they need not be distinct). To each i let R_i' be an extension of R_i to $I(S)$, that orders the elements in each R_i-slot trivially. Then there exist linear extensions R_i'' of R_i such that*

$$\bigcap R_i' = \bigcap R_i'',$$

that is, $x < y$ (R_i') for all $i \Leftrightarrow x < y$ (R_i'') for all i.

Proof. For $i = 2, \ldots, t$ let Q_i be a linear order extending R_i' and let P_i be the opposite ordering.

For $i = 1, \ldots, t-1$ define R_i'' by ordering the elements in each slot of R_i' by P_{i+1}. This defines R_i'' by Lemma 3 and R_i'' is clearly linear. Now let P_1 be the opposite of R_1''. Define R_t'' by ordering the elements in each slot by P_1.

Assume now that $x < y(R_i'')$ for all i but $x \not< y(R_j')$. Then $x, y \in L(S)$. Now no R_k' can place x in a greater slot than y, for then it would follow that $x > y(R_k'')$. Obviously x and y are in the same slot in R_j but they cannot be in distinct slots in any other R_k. For if so, let k be the first index in the sequence $j+1,\ldots, t, 1,\ldots j-1$ for which this is the case. Then for its predecessor i ($i=j$ if $k=j+1$)$x>y(R_i'')$ again contradicting the assumption. So, in particular, x and y are in the same slot of R_1 and R_t. But then if $x<y(R_1'')$, $x>y(R_t'')$ contradicting the hypothesis.

Definition. Let $R_1',\ldots,R_t'$ be a set of orders on $I(S)$ such that each R_i' restricts to a linear order R_i on S, and orders the elements of $L(S)$ in each R_i-slot trivially. We shall call the set $\{R_1',\ldots,R_t'\}$ an *interval base* if $\bigcap R_i'=I(S)$.

By the previous proposition dim $I(S)$ is the minimum cardinality of an interval base.

We now ask what conditions a set of linear orders on S must satisfy in order that they can be extended to an interval case on $I(S)$.

Proposition 5. *Let $\{R_i'\}$ be an interval base (or base) on $I(S)$ and let $x=(D,U)\in L(S)$. There exist i and j such that*

(i) *$(D, S\backslash D)$ is a slot of R_i' and x is placed in that slot by R_i',*
(ii) *$(S\backslash U, U)$ is a slot of R_j' and x is placed in that slot by R_j'.*

Clearly, if $D\cup U\neq S$, i and j must be distinct.

Proof. We shall prove only (i). Suppose there exists no such i. Let $y=(\emptyset, S\backslash D)$. Then x and y are incomparable in $L(S)$. However in each R_i' $x>s(R_i')$ for some $s\in S\backslash D$ and hence $x>y(R_i')$ for all i.

We shall say that x is *minimally placed* if it is placed in $(D, S\backslash D)$ and *maximally placed* if it is placed in $(S\backslash U, U)$.

Proposition 6. *Let $\{R_i\}$ be a set of at least two not necessarily distinct linear orders on S. A necessary condition that they can be extended to an interval base (or base) on $I(S)$ is the following:*

(C) *For any location (D, U) of $L(S)$ there exist distinct i and j such that D is a down set of R_i and U is an up set of R_j.*

Proof. If $D\cup U\neq S$ this is an immediate corollary of Proposition 5. If $D\cup U=S$, then D is a down set and U is an up set for all R_i.

The condition is not sufficient because it can be satisfied by two orders on any chain and yet dim $I(C_n)\to\infty$ by the results of [1]. More explicity, let x be the subset of $I(C_4)$ obtained by adjoining $\{a=(\emptyset,\{3,4\}),\ b=(\{1,2\},\emptyset),\ c=(\{1\},\{4\})\}$ to C_4. This set has the Hasse diagram shown in Fig. 1. Thus it is one of Trotter's minimal sets of dimension 3 [9]. It is clear that the set $\{R_1, R_2\}$ with R_i the natural order on C_4 satisfies the condition of Proposition 5. But it is easy to check that in

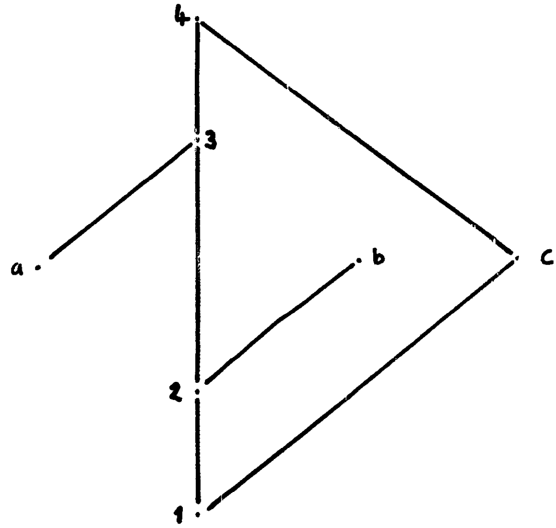

Fig. 1

any antichain extensions R_1' and R_2' we have $a<b$, $a<c$ or $c<b$ for both R_1' and R_2'.

I have not been able to find any stronger conditions for general sets S. When S is a chain the arguments of [1] give very strong results. Here we shall make the dual restriction, namely that the height of $S \leqslant 1$.

3. Sets of height $\leqslant 1$

We recall that the height of S is the maximum length n of a chain $s_0 < \cdots < s_n$ in S. Thus S has height 1 iff every element is maximal or minimal. In that case condition (C) is also sufficient if at least three orders are involved.

Theorem 7. *Let S have height $\leqslant 1$ and let $R_1, \ldots, R_k$, $k \geqslant 3$ be a set of orders on S satisfying condition (C). Then $R_1, \ldots, R_k$ can be extended to an interval base (and hence a base) on $I(S)$.*

Proof. Let A be the set of elements of S that are not minimal. B the set of those that are not maximal and C the rest. Divide $L(S)$ into three sets H, L and M (high, low and middle).

$$H = \{(D, \emptyset) \mid D \cap (A \cup C) \neq \emptyset\},$$
$$L = \{(\emptyset, U) \mid U \cap (B \cap C) \neq \emptyset\},$$
$$M = \{(D, U) \mid D \subseteq B,\ U \subseteq A\}.$$

Note that this partitions $L(S)$ into three antichains, and that every element of

M is incomparable to all elements of C. For each $x=(D, U)\in M$ choose $i=i(x)\neq j=j(x)$ such that D is a down set of R_i and U is an up set of R_j. Now construct extensions of $R_1, \ldots, R_k$ as follows.

(1) If $x\in H$ place x in the lowest admissible slot.

(2) If $x\in L$ place x in the highest admissible slot.

(3) If $x\in M$, for $i=i(x)$ place x in the slot $(D, S\backslash D)$; for $j=j(x)$ place x in the slot $(S\backslash U, U)$; otherwise place x in the slot immediately below the lowest element of A. That is, x is minimally placed in R_i', maximally placed in R_j' and "centrally placed" in every other order.

Order the elements in each slot trivially.

We claim the orders $R_i' i=1,\ldots,k$ form an interval base. So assume $x\not\leqslant y(I(S))$, we must show $x\not\leqslant y(R_i)$ for some i.

Case 1. x, $y\in S$. Let $U=\{t\in S \mid t\geqslant x\}$. Then U is an up set in some R_i. But $y\notin U$ so $x\not\leqslant y(R_i)$.

Case 2. $X\in M$. Let $x=(D, U)$ and $j=j(x)$. Then $x\leqslant y(R_j')\Rightarrow x\leqslant y\ (I(S))$ or $y\in M$, $y=(D', U')$, $U'\subsetneq U$ and $j=j(y)$. Now consider $i=i(y)$. Then $x\leqslant y(R_i')\Rightarrow D\subsetneq D'$ and $i=i(x)$. Now in any third order R_n', x and y are placed in the same slot, so $x\not\leqslant y(R_n')$.

Obviously the same argument holds if $y\in M$. We may now assume that neither x nor y is in M.

Case 3. $x=(\emptyset, U)\in L$ (or dually $y\in H$). Let U be an upset of, say, R_i. Then $x\leqslant y(R_1')\Rightarrow x\leqslant y(I(S))$ or $y\in L$, $y=(\emptyset, U')$ and $U'\subsetneq U$. Now $D=U'\cap(B\cup C)\neq\emptyset$ and it is a down set of S since all elements of $B\cup C$ are minimal. Hence D is a down set of some R_i say R_2. But then x and y or both placed in the slot $(\emptyset, S)$ in R_2 thus $x\not\leqslant y(R_2)$.

Case 4. $x=(D, \emptyset)\in H$ (or dually $y\in L$). Then $U=D\cap(A\cup C)\neq\emptyset$ is an upset of S and hence of some R_i say R_1. But then x is maximal in R_1 so $x\not\leqslant y(R_1)$ for all $y\neq x$ in $I(S)$.

We now specialise even further to the case when $C=\emptyset$ and all elements of A lie above all elements of B. In that case we can calculate $\dim I(S)$ explicitly.

Definition. The *double antichain* $D(m, n) m\geqslant 0$, $n\geqslant 0$ is the union of two antichains A and B with $|A|=m$, $|B|=n$ and all elements of A below all elements of B.

These double antichains are the sets that determine the cardinality of $U(S)$. In [4] it is shown that $\dim(U(S))$ is the maximum cardinality of a double antichain subset of S.

For our final theorem we denote by $\lceil r\rceil$ the ceiling of r, that is the smallest integer not less than r (the floor of r, $\lfloor r\rfloor$ is defined dually), and by $p(n)$ the width of the power set of a set of n elements, that is the binomial coefficient. $\binom{n}{\lfloor n/2\rfloor}$.

Theorem 8. *Let $S = D(m, n)$ be a double antichain with $m \leqslant n$.*
If $m+n \geqslant 4$, then $\dim I(S) = \max\{p(n), \lceil \frac{2}{3}(p(m)+p(n)) \rceil\}$.
If $m+n = 2, 3$, then $\dim I(S) = m+n$.
If $m+n = 0, 1$, then $\dim I(S) = m+n+1$.

Proof. Let the constituent antichains be A, B, with $|A| = m$, $|B| = n$ and all elements of A less than all elements of B. We first use Proposition 6 to establish a lower bound for $\dim I(S)$. If $R_1, \ldots, R_u$ are a set of linear orders of S that can be extended to a base of $I(S)$ then every subset of A must occur as a down set and every subset of B must occur as an up set in some order R_i. We consider the subsets of A of order $\lfloor m/2 \rfloor$, $A_1, \ldots, A_{p(m)}$ and those of B of order $\lfloor n/2 \rfloor$, $B_1, \ldots, B_{p(n)}$. Unless $S = \emptyset$ in which case the theorem is trivial, $A_i \cup B_j \neq S$ so it must be possible to choose the orders for A_i and B_j distinct from each other. Now if B_i is an up set of R_j, B_k cannot also be an up set of R_j for $k \neq i$. Thus

$$\dim I(S) \geqslant p(n).$$

Now call a set A_i a "single" if there is a unique $k = 1, \ldots, u$ such that A_i is a down set of R_k. A "single" is defined dually for the B_i's. Let there be s_1 singles among the A_i's and s_2 singles among the B_1's. Then the other sets among the A_i occur in at least two R_k. Thus

$$u \geqslant s_1 + 2(p(m) - s_1)$$

and similarly

$$u \geqslant s_2 + 2(p(n) - s_2).$$

Further, since the orders R_i form a base, Proposition 6 implies that two singles A_i and B_j are never a down set and an up set of the same R_k. Thus

$$u \geqslant s_1 + s_2.$$

Hence

$$3u \geqslant 2(p(m) + p(n)).$$

This establishes a lower bound for $\dim I(S)$ which is in fact attained for all sets $S = D(m, n)$ with $m+n \geqslant 2$, except $D(1, 2)$. $I(D(1, 2))$ contains the set shown in Fig. 2, which is one of Trotter's examples of minimal sets of dimension 3, [9].

We now construct a set of linear orders $\{R_i\}$ on S satisfying condition (C) and with $\max\{p(n), \lceil \frac{2}{3}(p(m)+p(n)) \rceil\}$ elements. Notice that if D is a down set of S then $D \subseteq A$ or $S \backslash D \subseteq B$ and in the latter case the only location (D, U) in $I(S)$ is $(D, \emptyset)$. Thus in verifying (C) we need only consider pairs (S, T) with $S \subseteq A$, $T \subseteq B$.

Now the linear orders R_i on S correspond uniquely to the pairs of maximal chains (C_i, C_i') in the power sets $P(A)$ and $P(B)$. (the chains C_i consist of the down sets of R_i in A and the chains C_i' of the up sets of R_i in B). Clearly if (C) is to be satisfied for $R_1, \ldots, R_u$ corresponding to $(C_1, C_i'), \ldots, (C_u, C_u')$, then the

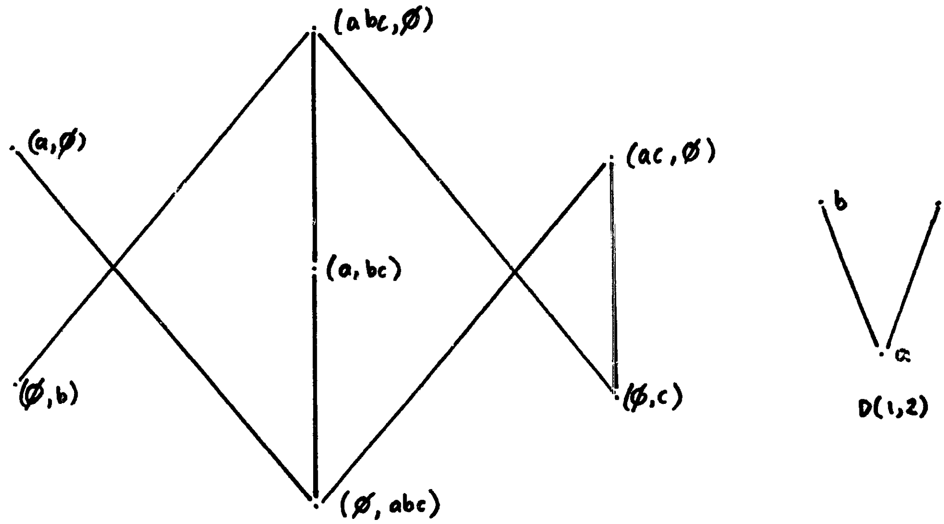

Fig. 2.

chains C_i and C_i' must cover $P(A)$ and $P(B)$. By Sperner's Theorem [6] and Dilworth's Theorem [3] it is possible to cover $P(A)$ by chains $C_1, \ldots, C_{p(m)}$ and $P(B)$ by chains $C_1', \ldots, C_{p(n)}'$. Choose such coverings enlarging the chains if necessary to maximal chains, so that any pair (C_i, C_j') defines a linear order on S.

Case 1. $2p(m) \leqslant p(n)$. This is the case if $m \leqslant n-2$ or $m = n-1$ and n is even. In this case

$$p(n) \geqslant \tfrac{2}{3}(p(m)+p(n)).$$

For $i = 1, \ldots, p(n)$ put $C_i = C_r$, where r is the remainder left by i on division by $p(m)$. Let the orders R_i correspond to (C_i, C_i') for $i = 1, \ldots, p(n)$. These orders clearly satisfy (C), because for any $S \subseteq A$, $S \in C_i$ for at least two distinct values of i.

Case 2. $p(m) \leqslant p(n) < 2p(m)$. This occurs if $m = n$ or $m = n-1$ and n is odd. Let $u = \lceil \tfrac{2}{3}(p(m)+p(n)) \rceil$. Then

$$p(n) < u < 2p(m).$$

Let

$$d_1 = u - p(m), \qquad s_1 = p(m) - d_1,$$
$$d_2 = u - p(n), \qquad s_2 = p(n) - d_2.$$

Then $d_1, d_2, s_1, s_2 > 0$. Furthermore

$$s_1 + s_2 = 2(p(m)+p(n)-u) \leqslant u.$$

Now define for $i = 1, \ldots, d_1$ and $j = 1, \ldots, d_2$

$$C_{p(m)+i} = C_{s_1+i}, \qquad C'_{p(n)+j} = C'_{s_2+j}.$$

For $i = 1, \ldots u$ let R_i correspond to (C_i, C'_{u+1-i}). To show that $R_1, \ldots, R_u$ satisfy

(C) choose $S \subseteq A$ and $T \subseteq B$. Let $S \in C_i$ and $T \in C_j'$ ($i \leq p(m)$, $j \leq p(n)$). Then S is a down set of R_i and T is an upset of R_k, $k = u + 1 - j$. If $k \neq i$, all is well. On the other hand if $k = i$, then $i + j = u + 1 > s_i + s_2$, so $i > s_1$ or $j > s_2$, say $i > s_1$. But then $C_i = C_{p(m)+i-s_1} = C_l$ and so A is a down set of R_l with $l \neq i$ and again condition (C) is satisfied.

Now condition (C) is sufficient for extendability to a base provided the number of orders involved is at least 3, which is the case whenever $m \geq 2$ or $n \geq 3$. Thus the dimension of $I(D(m, n))$ is equal to the bound except possibly when $m \leq 1$ and $n \leq 2$. For $S = D(1, 2)$ we already know that $\dim I(S) \geq 3$, but we can satisfy (C) with two linear orders. Adjoining an arbitrary third linear order we see that $\dim I(S) \leq 3$ and hence $\dim I(S) = 3$. For $m + n \leq 2$ the values are easily found by inspection.

Corollary. *If S is an antichain of n elements,* $\dim I(S) = p(n)$.

It is tempting to try and extend this technique to the case when A and B are not necessarily antichains, establishing bounds for $\dim (I(S))$ in terms of $\dim I(A)$ and $\dim I(B)$, where S is the set "B over A". Unfortunately the non-sufficienty of the condition in Proposition 6 causes severe difficulties as it is now no longer sufficient to place locations high in one order and low in another as the example of Fig. 1 shows. The arguments of [1] do however offer some hope that stronger techniques may exist to deal with that case.

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